THE POINT-ARBORICITY OF A GRAPH(1)

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ABSTRACT

The point-arboricity $\rho(G)$ of a graph G is defined as the minimum number of subsets in a partition of the point set of G so that each subset induces an acyclic subgraph. Dually, the tulgeity $\tau(G)$ is the maximum number of disjoint, point-induced, non-acyclic subgraphs contained in G. Several results concerning these numbers are presented, among which are formulas for the point arboricity and tulgeity of the class of complete n-partite graphs.

Introduction. The *arboricity* of a graph G is defined as the minimum number of subsets into which the line set of G can be partitioned so that each subset induces an acyclic subgraph. The arboricity of the complete graphs and complete bipartite graphs has been studied by Beineke [1] and that of graphs in general by Nash-Williams [6, 7]. Dual to the concept of arboricity is the maximum number of line-disjoint subgraphs contained in G so that each subgraph is not acyclic. Since each such subgraph may be assumed a cycle, this number is referred to as the *cycle multiplicity* of G. A formula for the cycle multiplicity of complete graphs and complete bipartite graphs is given in [2].

Analogous to the numbers just described are their point versions, as discussed in [2], which are dual in the same sense as arboricity and cycle multiplicity. The *point-arboricity* $\rho(G)$ of a graph G is the least number of subsets into which the point set of G can be divided so that every subset induces an acyclic subgraph. Clearly, $\rho(G) \geqq 1$ for every nonempty graph G and $\rho(G) = 1$ if and only if G itself is acyclic. It is the main object of this article to investigate the concept of point-arboricity. In addition, however, we consider the dual topic of the *tulgeity* or *point-cycle multiplicity* $\tau(G)$ of a graph G, defined as the maximum number of disjoint, point-induced subgraphs contained in G so that no subgraph is acyclic. Of course, $\tau(G) = 0$ if and only if G is acyclic. Equivalently, $\tau(G)$ is the maximum number of disjoint cycles in G. All graphs considered in this paper are finite and contain no loops and no multiple edges.

Basic Results. We begin by presenting a few elementary results concerning point-arboricity and tulgeity.

A subgraph H of a graph G is *point-induced* if every line of G which joins

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(1) Definitions not given in this article may be found in [5].

two points of H is also a line of H. A subgraph is *line-induced* if it contains no isolated points.

A maximal connected subgraph is a *component* of G while a maximal connected subgraph containing no *cutpoints* is a *block* of G. Since every cycle of G is necessarily contained in a block and hence in a component of G, we arrive at the following observation.

THEOREM 1. *For any graph G*
(i) $\rho(G) = \max \rho(C) = \max \rho(B)$ *and*
(ii) $\tau(G) = \sum \tau(C) \leqq \sum \tau(B)$,

the maxima and sums being taken over all components C and blocks B of G, respectively.

The *chromatic number* $\chi(G)$ is the minimum number of subsets in a partition of the point set of G such that each subset induces a subgraph containing no lines. Each such subset is called a *color class*. Since each color class is acyclic and every acyclic graph is 2-colorable, it follows that $\rho(G) \leqq \chi(G) \leqq 2\rho(G)$ for every graph G. The point-arboricity may also be interpreted as a "coloring number," since $\rho(G)$ is the minimum number of colors needed to color the points of G so that no cycle has all its points colored the same.

We now present an upper bound for $\rho(G)$ in terms of the maximum degree of the points of G. By $\{x\}$ is meant the least integer not less than x.

THEOREM 2. *If* $\max\deg G$ *denotes the maximum degree of the points of G, then*

$$\rho(G) \leqq \left\{\frac{1 + \max\deg G}{2}\right\}.$$

Proof. We proceed by induction on the number p of points of G, the result being obvious for $p = 1$. We thus assume the formula holds for all graphs H having $p - 1$ points, $p \geqq 2$, and let G be a graph with p points. Select a point v of G, and consider the graph $G - v$ obtained from G by deleting v and all lines incident with v. By hypothesis,

$$\rho(G - v) = r \leqq \left\{\frac{1 + \max\deg(G - v)}{2}\right\}.$$

Hence it is possible to partition the point set $V - \{v\}$ of $G - v$ into subsets $V_1, V_2, \cdots, V_r$ so that each V_i induces an acyclic subgraph. If

$$r < \left\{\frac{1 + \max\deg G}{2}\right\},$$

then the partition $V_1, V_2, \cdots, V_r, \{v\}$ produces the desired result. On the other hand, if $r = \left\{\frac{1 + \max\deg G}{2}\right\}$, at least one of the subsets V_i, $1 \leqq i \leqq r$, contains at most one point adjacent with v. If V_j is such a set, then each of the r subsets $V_1, V_2, \cdots, V_j \cup \{v\}, \cdots, V_r$ induces an acyclic subgraph, completing the proof.

The preceding result is best possible since the upper bound is the actual value of the point-arboricity in the case of complete graphs. For the class of planar graphs this bound is not particularly useful since such graphs may contain points of high degree. However, we have already observed that $\rho(G) \leqq \chi(G)$ so that by the Five Color Theorem, $\rho(G)$ is certainly no greater than 5 for planar graphs G and probably no larger than 4. This leads us to our next result, which is also discussed in [2].

THEOREM 3. *For every planar graph G,*

$$\rho(G) \leqq 3.$$

Proof. We proceed by induction on the number p of points of G, the result holding trivially for $p = 1$.

Assume $\rho(G') \leqq 3$ for all planar graphs G' having $p \geqq 1$ points, and let G be a planar graph with $p + 1$ points. Since G is planar, it contains a point v of degree five or less. The graph $G - v$ obtained from G by removing v (and all lines incident with v) is planar and has p points; therefore $\rho(G - v) \leqq 3$. Thus, the point set of $G - v$ can be partitioned into three (not all necessarily nonempty) subsets V_1, V_2, V_3 such that each V_i induces an acyclic subgraph of $G - v$. Since the degree of v does not exceed 5, at least one of the subsets V_i contains at most one point which is adjacent with v. Let V_1 be such a set. Then $V_1 \cup \{v\}, V_2, V_3$ constitutes a partition of the point set of G such that each subset induces an acyclic subgraph. Hence $\rho(G) \leqq 3$.

Of course, for all planar graphs G for which $\rho(G) \leqq 2$, it follows that $\chi(G) \leqq 4$; therefore a search for a planar graph which would disprove the Four Color Conjecture must be made among those planar graphs having point-arboricity 3.

A *subdivision* of a graph G is a graph obtained from G by replacing some line $x = uv$ of G by a new point w and the two new lines uw and wv. Two graphs G_1 and G_2 are *homeomorphic* if there exists a pair of isomorphic graphs G_1' and G_2' such that G_i' can be obtained from G_i, $i = 1,2$, by a sequence of subdivisions. If G' is obtained from G by subdivision, then clearly $\rho(G') \leqq \rho(G)$.

If every line of a graph G is subdivided, then a bipartite graph G' results so that $\chi(G') = 2$. Thus $\rho(G') \leqq 2$. Since G and G' are homeomorphic, we see that in general homeomorphic graphs need not have the same point-arboricity. It is, however, a routine matter to establish the following result.

REMARK. If G_1 and G_2 are homeomorphic, then $\tau(G_1) = \tau(G_2)$.

Bounds for $\tau(G)$ have been obtained by Corrádi and Hajnal [3] and by Dirac and Erdös [4]; in particular, it was shown in [3] that if G is a graph with $p \geqq 3k$ points having $\min\deg G \geqq 2k$, then $\tau(G) \geqq k$.

The Point-Arboricity of Complete n-Partite Graphs. The *complete n-partite graph* $K(p_1, p_2, \cdots, p_n)$ has its point set V partitioned into n disjoint nonempty subsets V_i, where $|V_i| = p_i$, such that a line joins two points u and v if and only if $u \in V_j$ and $v \in V_k$, where $j \neq k$. If $p_i = 1$ for every i, the graph is *complete.* If $n = 2$ the graph is called *complete bipartite.*

As was mentioned earlier, one of the major problems dealing with arboricity and cycle multiplicity has been the determination of these numbers for the complete graphs and the complete bipartite graphs. We now consider the question of finding the point-arboricity and tulgeity of the complete n-partite graphs. We begin with point-arboricity.

Since $\chi(K(p_1, p_2, \cdots, p_n)) = n$, we know that $\rho = \rho(K(p_1, p_2, \cdots, p_n)) \leqq n$. The amount by which ρ and n differ is given next.

THEOREM 4. *The point-arboricity of the complete n-partite graph* $K(p_1, p_2, \cdots, p_n)$, $1 \leqq p_1 \leqq p_2 \leqq \cdots \leqq p_n$, *is given by*

$$\rho(K(p_1, p_2, \cdots, p_n)) = n - \max\left\{k \,\middle|\, \sum_0^k p_i \leqq n - k\right\},$$

where we define $p_0 = 0$.

Proof. We employ induction on n. For $n = 1$, $\rho(K(p_1)) = 1$ follows immediately.

Assume the formula holds for n, $n \geqq 1$, and consider the graph $K(p_1, p_2, \cdots, p_{n+1})$ with subsets $V_1, V_2, \cdots, V_{n+1}$ as described in the definition. For the subgraph $K(p_1, p_2, \cdots, p_n)$, suppose that $\sum_0^t p_i \leqq n - t$ but that $\sum_0^{t+1} p_i > n - (t+1)$. By hypothesis, then $\rho(K(p_1, p_2, \cdots, p_n)) = n - t$. Since $K(p_1, p_2, \cdots, p_n)$ is a subgraph of $K(p_1, p_2, \cdots, p_{n+1})$, $\rho(K(p_1, p_2, \cdots, p_n)) \leqq \rho(K(p_1, p_2, \cdots, p_{n+1}))$. Also, since the additional set of p_{n+1} points used in forming $K(p_1, p_2, \cdots, p_{n+1})$ induces an acyclic subgraph, it follows that $\rho(K(p_1, p_2, \cdots, p_{n+1})) \leqq \rho(K(p_1, p_2, \cdots, p_n)) + 1$. We now consider two cases.

Case 1. Suppose $\sum_0^{t+1} p_i > (n+1) - (t+1) = n - t$. This implies that $(n+1) - \max\{k \mid \sum_0^k p_i \leqq (n+1) - k\} = n + 1 - t$. Thus in this case we wish to show that $\rho(K(p_1, p_2, \cdots, p_{n+1})) = \rho(K(p_1, p_2, \cdots, p_n)) + 1$. Assume this is not the case so that $\rho(K(p_1, p_2, \cdots, p_{n+1})) = \rho(K(p_1, p_2, \cdots, p_n))$. The complete $(n+1)$-partite graph $K(p_1, p_2, \cdots, p_t, p_{t+1}, \cdots, p_{t+1}) = K'$ is a subgraph of $K(p_1, p_2, \cdots, p_{n+1})$; therefore, $\rho(K') \leqq \rho(K(p_1, p_2, \cdots, p_{n+1})) = \rho(K(p_1, p_2, \cdots, p_n)) = n - t$. However, K' contains $\sum_0^{t+1} p_i + (n-t)p_{t+1}$ points, which implies that in any partition of the point set of K' into $n - t$ (or fewer) subsets, at least one such subset must contain at least $(\sum_0^{t+1} p_i + (n-t)p_{t+1})/(n-t)$ points. However, $\sum_0^{t+1} p_i > n-t$ so that one of the subsets of the partition contains at least $p_{t+1}+2$ points. It is now easy to see that such a subset induces a subgraph containing a triangle if $p_{t+1}=1$ or a 4-cycle if $p_{t+1}>1$. This is a contradiction.

Case 2. Suppose $\sum_0^{t+1} p_i \leqq n-t$.

Since $\sum_0^{t+1} p_i > n-t-1$, it follows directly that $\sum_0^{t+2} p_i > n-t-1$ so that in this case $(n+1) - \max\{k \mid \sum_0^k p_i \leqq (n+1)-k\} = (n+1)-(t+1) = n-t$. Therefore we wish to show here that $\rho(K(p_1,p_2,\cdots,p_{n+1})) = \rho(K(p_1,p_2,\cdots,p_n))$. Since $\sum_0^{t+1} p_i \leqq n-t$ or, equivalently, since $|V_1 \cup V_2 \cup \cdots \cup V_{t+1}| \leqq |\{V_{t+2}, V_{t+3}, \cdots, V_{n+1}\}|$ we may add one element from $V_1 \cup V_2 \cup \cdots \cup V_{t+1}$ to each of the sets $V_{t+2}, V_{t+3}, \cdots, V_s$, where $s \leqq n+1$ so as to exhaust $V_1 \cup V_2 \cup \cdots \cup V_{t+1}$. Since each set $V_j \cup \{u\}$, $t+2 \leqq j \leqq s$, induces the acyclic subgraph $K(1,p_j)$, we have $\rho(K(p_1,p_2,\cdots,p_{n+1})) \leqq n-t$ which implies that $\rho(K(p_1,p_2,\cdots,p_{n+1})) = n-t$.

The point-arboricity of the complete graphs and complete bipartite graphs can now be given.

COROLLARY 4a. *For the complete graph K_p with p points,*

$$\rho(K_p) = \left\{\frac{p}{2}\right\},$$

while for the complete bipartite graph $K(p_1,p_2)$, $p_1 \leqq p_2$,

$$\rho(K(p_1,p_2)) = \begin{cases} 2 & \text{if } p_1 > 1 \\ 1 & \text{if } p_1 = 1. \end{cases}$$

The Tulgeity of Complete n-Partite Graphs. In this section we derive a formula for the tulgeity of the complete n-partite graphs.

Since $\tau(G)$ is the maximum number of disjoint cycles in G and since every cycle contains at least three points, an obvious upper bound is obtained.

REMARK. For any graph G with p points, $\tau(G) \leqq [p/3]$.

A result which will be useful in the proof of Theorem 5 to follow involves the concept of a maximum matching. A *maximum matching* in a graph G is a maximum set M of lines so that no two lines of M are adjacent. The following formula was presented in [2].

THEOREM. *If M is a maximum matching in the complete n-partite graph $K(p_1,p_2,\cdots,p_n)$, $1 \leqq p_1 \leqq p_2 \leqq \cdots \leqq p_n$, and $\sum p_i = p$, then*

$$|M| = \min\left(\sum_0^{n-1} p_i, [p/2]\right),$$

where $p_0 = 0$.

We now present a formula for the tulgeity of the complete n-partite graphs.

THEOREM 5. *For the complete n-partite graph $G = K(p_1, p_2, \cdots, p_n)$, $1 \leqq p_1 \leqq p_2 \leqq \cdots \leqq p_n$, and $\sum p_i = p$,*

$$\tau(G) = \min\left(\left[\frac{1}{2}\sum_{0}^{n-1} p_i\right], [p/3]\right),$$

where $p_0 = 0$.

Proof. We begin with a few preliminary observations. Any point-induced subgraph of $K(p_1, p_2, \cdots, p_n)$ which is a cycle necessarily contains points from at least two of the subsets $V_1, V_2, \cdots V_n$. If such a cycle contains points from exactly two of the subsets V_i, then the cycle has precisely two points from each of the two subsets in which case a 4-cycle results. The only other possibility is a triangle which contains a point from each of three different subsets V_i. Thus for $K(p_1, p_2, \cdots, p_n)$ we wish to determine the maximum of disjoint triangles and 4-cycles.

It is obvious that $\tau(K(p_1, p_2)) = [p_1/2]$ since $K(p_1, p_2)$ has no triangles.

The maximum number of disjoint triangles in $K(p_1, p_2, p_3)$ is clearly p_1. The deletion of p_1 points from each of V_1, V_2 and V_3 results in a graph which has a maximum of $[\frac{1}{2}(p_2 - p_1)]$ disjoint 4-cycles. Thus, $\tau(K(p_1, p_2, p_3)) \geqq p_1 + [\frac{1}{2}(p_2 - p_1)] = [\frac{1}{2}(p_1 + p_2)]$, but since every cycle contains at least two points not in V_3, $\tau(K(p_1, p_2, p_3)) \leqq [\frac{1}{2}(p_1 + p_2)]$. Therefore $\tau(K(p_1, p_2, p_3)) = [\frac{1}{2}(p_1 + p_2)]$.

For $n \geqq 4$, we use induction on p and consider 3 cases, the formula being evident for small values of p.

Case 1. Suppose $\sum_1^{n-2} p_i \leqq p_{n-1}$.

Since $\sum_1^{n-2} p_i \leqq p_{n-1}$, a maximum matching for $K(p_1, p_2, \cdots, p_{n-1})$ contains $\sum_1^{n-2} p_i$ lines. These lines together with $\sum_1^{n-2} p_i$ points in V_n determine $\sum_1^{n-2} p_i$ disjoint triangles in $G = K(p_1, p_2, \cdots, p_n)$. From the remaining points in V_{n-1} and V_n, $[\frac{1}{2}(p_{n-1} - \sum_1^{n-2} p_i)]$ disjoint 4-cycles are determined. Therefore $\tau(G) \geqq \sum_1^{n-2} p_i + [\frac{1}{2}(p_{n-1} - \sum_1^{n-2} p_i)] = [\frac{1}{2}\sum_1^{n-1} p_i]$. Since every cycle contains at least two points not in V_n, $\tau(G) \leqq [\frac{1}{2}\sum_1^{n-1} p_i]$. Thus, $\tau(G) = [\frac{1}{2}\sum_1^{n-1} p_i]$.

Case 2. Suppose $p \leqq 3p_n$ and $\sum_1^{n-2} p_i > p_{n-1}$.

In this case a maximum matching in $K(p_1, p_2, \cdots, p_{n-1})$ contains $[\frac{1}{2}\sum_1^{n-1} p_i]$ lines. Since $[\frac{1}{2}\sum_1^{n-1} p_i] \leqq p_n$ there are sufficiently many points in V_n to determine $[\frac{1}{2}\sum_1^{n-1} p_i]$ disjoint triangles implying that $\tau(G) \geqq [\frac{1}{2}\sum_1^{n-1} p_i]$. Now $\tau(G) = [\frac{1}{2}\sum_1^{n-1} p_i]$, for as before, we always have the inequality $\tau(G) \leqq [\frac{1}{2}\sum_1^{n-1} p_i]$.

Case 3. Suppose $p > 3p_n$.

In this case, we assume the formula holds for all complete n-partite graphs with less than p points. We select a triangle in $K(p_1, p_2, \cdots, p_{n-1})$ and remove a point from each of the three subsets involved. We then relabel the subsets as

$V_1', V_2', \cdots, V_n'$, where $|V_i'| = p_i'$ so that $p_n' = p_n$ and $\sum_1^{n-1} p_i' = \sum_1^{n-1} p_i - 3$. Some of the sets V_i' may be empty. By the inductive hypothesis, $\tau(K(p_1', p_2', \cdots, p_n')) = \min([\frac{1}{2} \sum_1^{n-1} p_i'], [\frac{1}{3} \sum_1^n p_i']) = \min([\frac{1}{2}(\sum_1^{n-1} p_i - 3)], [p/3 - 1])$. This implies that $\tau(G) \geqq 1 + \min([\frac{1}{2} \sum_1^{n-1} p_i - 3)], [p/3 - 1]) = \min([\frac{1}{2}(\sum_1^{n-1} p_i - 1)], [p/3])$. From $p > 3p_n$ it follows that $[\frac{1}{2}(\sum_1^{n-1} p_i - 1)] \geqq [p/3]$. This completes the proof.

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